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Do anyons solve Heisenberg's Urgleichung in one dimension

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Abstract. We construct solutions to the chiral Thirring model in the framework of algebraic quantum field theory. We find that for all positive temperatures there are fermionic solutions only if the coupling constant is $\lambda = \sqrt{2(2n+1)\pi}$, $n \in \mathbf{N}$.

1 Introduction

It is usually taken for granted that fermions should enter the basic formalism of the fundamental theory of elementary particles, the ultimate version of this opinion being Heisenbergs "Urgleichung" [1], in which no bose fields are present at all. The opposite point of view, namely that theory including only observable fields, necessarily uncharged bosons, is capable of describing evolution and symmetries of a physical system, is the kernel of algebraic approach to QFT, due to Haag and Kastler [2]. Actually, the question which is thus posed and which is of principal importance is whether and in which cases definite conclusions about the time evolution and symmetries of charged fields can be drawn from the knowledge about the observables that is gained through experiment. Furthermore, before claiming that an "Urgleichung" of the type

$$\partial\!\!\!/\psi(x) = \lambda\psi(x)\bar{\psi}(x)\psi(x) \qquad (1)$$

determines the whole Universe one should see whether it determines anything mathematically.

Two–dimensional models offer a possibility to get a better feeling for these problems due to the bose–fermi duality which takes place in two–dimensional spacetime. This phenomenon amounts to the fact that in certain models formal functions of fermi fields can be written that have vacuum expectation values and statistics of bosons and vice versa. The equivalence is understood within perturbation theory: the perturbation series for the so–related theories are term–by–term equivalent (they may perfectly well exist even if the models are not exactly solvable or if their physical sensibility is doubtful).

There are two facts which make such a duality possible. First comes the main reason why soluble fermion models exist in two–dimensions, that is that fermion currents can be constructed as "fields" acting on the representation space for the fermions. Also, the "bosons into fermions" programme rests on the fact that bosons in question are just the currents and fermions are essentially determined by their commutation relations with them. Second comes the observation which has been made in the pioneering works by Jordan [3] and Born [4]: due to the unboundedness from below of the free–fermion Hamiltonian the fermion creation and annihilation operators must undergo what we should call now a Bogoliubov transformation which in addition leads to the appearance of an anomalous term (later called "Schwinger term") in the current commutator, that in turn actually enables the "bosonization".

The "fermions into bosons" part of the bose–fermi duality is fairly well established, so that consistent expressions exist for the fermion bilinears that are directly related to the observables of the theory.

The problem of rigorous definitions of operator–valued distributions and eventually operators having the basic properties of fermion fields by taking functions of bosonic fields is rather more delicate. On the level of operator valued distributions solutions have been given by Dell'Antonio et al. [5] and Mandelstam [6] and on the level of operators in a Hilbert space – by Carey and collaborators [7,8] and in a Krein space by Acerbi, Morchio and Strocchi [9].

Our goal is to see what elements are needed to make a solution of an equation of the type (1) well defined. We shall not only reduce it to (1+1) dimensions but will consider only one chiral component (a left or right mover)

[a] *Dedication.* F. Schwabl is well–known for his contributions in condensed matter physics and his book on quantum mechanics. However he was also among the pioneers for solving (1+1)–dimensional quantum field theories and it is with pleasure that we dedicate this note to his 60^{th} birthday.

$\psi(x)$, where x stands for $t \pm x$. Thus the question is how one can give a precise meaning to the three ingredients

$$\begin{aligned} &(a)\ [\,\psi^*(x), \psi(x')\,]_+ = \delta(x-x'), && \\ &\qquad [\,\psi(x), \psi(x')\,]_+ = 0 && \text{CAR} \\ &(b)\ \tfrac{1}{i}\tfrac{d}{dx}\psi(x) = \lambda j(x)\psi(x) && \text{Urgleichung} \\ &(c)\ j(x) = \psi^*(x)\psi(x) && \text{Current} \end{aligned} \qquad (2)$$

Since (2b) involves derivatives of objects which are according to (2a) rather discontinuous it is expedient to pass right away to the level of operators in Hilbert space since there are plenty of topologies to control the limiting procedures. In general norm convergence can hardly be hoped for but we have to strive at least for strong convergence such that the limit of the product is the product of the limits. With $\psi_f = \int_{-\infty}^{\infty} dx f(x)\psi(x)$, (2a) becomes

$$[\,\psi_f^*, \psi_g\,]_+ = \langle f|g\rangle \qquad (3)$$

for $f \in L^2(\mathbf{R})$ and $\langle .|.\rangle$ the scalar product in $L^2(\mathbf{R})$. This shows that ψ_f's are bounded and form the C^*–algebra CAR. There the translations $x \to x+t$ give an automorphism τ_t and we shall use the corresponding KMS–states ω_β and the associated representation π_β to extend CAR. Though there $j = \infty$, one can give a meaning to j as a strong limit in $\mathcal{H}_\beta$ by smearing $\psi(x)$ over a region ε to $\psi_\varepsilon(x)$ and define

$$j_f = \int dx f(x) \lim_{\varepsilon\to 0} \left(\psi_\varepsilon^*(x)\psi_\varepsilon(x) - \omega_\beta(\psi_\varepsilon^*(x)\psi_\varepsilon(x))\right), \quad f : \mathbf{R} \to \mathbf{R}$$

These limits exist in the strong resolvent sense and define self–adjoint operators which determine with

$$e^{ij_f} e^{ij_g} = e^{\frac{i}{8\pi}\int dx (f(x)g'(x) - f'(x)g(x))} e^{ij_{f+g}} \qquad (4)$$

the current algebra $\mathcal{A}_c$. Its Weyl structure is the same for all $\beta > 0$ and ω_β extends to $\mathcal{A}_c$.

To construct the interacting fermions which on the level of distributions look like

$$\Psi(x) = Z\, e^{i\lambda \int_{-\infty}^{x} dx' j(x')}$$

(with some renormalization constant Z) poses two problems, one infrared and one ultraviolet. For

$$\Psi_{\varepsilon,R}(x) = e^{i\lambda \int dx' (\varphi_\varepsilon(x-x') - \varphi_\varepsilon(x-x'+R)) j(x')},$$

$$\varphi_\varepsilon(x) := \begin{cases} 1 & \text{for } x \le -\varepsilon \\ -x/\varepsilon & \text{for } -\varepsilon \le x \le 0 \\ 0 & \text{for } x \ge 0 \end{cases}$$

neither the limit $R \to \infty$ nor the limit $\varepsilon \to 0$ exist even as weak limits in $\mathcal{H}_\beta$. Thus one has to extend $\pi(\mathcal{A}_c)''$ to accomodate this kind of objects.

There are two equivalent ways of handling the infrared problem. Since the automorphism generated by the unitaries $\Psi_{\varepsilon,R}(x)$ converges to a limit γ for $R \to \infty$, one can form with it the crossed product $\bar{\mathcal{A}}_c = \mathcal{A}_c \overset{\gamma}{\rtimes} \mathrm{Z}$, so that in $\bar{\mathcal{A}}_c$ there are unitaries with the properties which the limit should have. On the other hand, the symplectic form in (4) and the state ω_β can be defined for the limiting element $\Psi_\varepsilon(x)$. This is what we will do in the text but we also follow the former route in Appendix B. In any case $\bar{\mathcal{H}}_\beta$ assumes a sectorial structure, the subspaces $\mathcal{A}_c \prod_{i=1}^{n} \Psi_\varepsilon(x_i)|\Omega\rangle$ for different n are orthogonal and thus may be called n–fold charged sectors. The $\Psi_\varepsilon(x)$'s have the property that for $|x_i - x_j| > 2\varepsilon$ they obey anyon statistics with parameter λ^2 and an Urgleichung (2b) where $j(x)$ is averaged over a region of lenght ε below x.

Removing the ultraviolet cut–off , $\varepsilon \downarrow 0$, one could proceed as before but in this case the sectors abound and the subspaces $\mathcal{A}_c\Psi(x)|\Omega\rangle$ become orthogonal for different x, so $\bar{\mathcal{H}}_\beta$ becomes non–separable. To get canonical fields of the type (3) one has to combine $\varepsilon \downarrow 0$ with a field renormalization $\Psi_\varepsilon \to \varepsilon^{-1/2}\Psi_\varepsilon$ such that

$$\lim_{\varepsilon\downarrow 0} \varepsilon^{-1/2} \int dx f(x) \Psi_\varepsilon(x) = \Psi_f$$

converge strongly in $\bar{\mathcal{H}}_\beta$ and satisfy (2b) in sense of distributions. However, they are not fermions but anyons and only for $\lambda = \sqrt{2(2n+1)\pi}$, $n \in \mathbf{N}$ they are fermions. Thus we find that there is indeed some magic about the Urgleichung inasmuch as on the quantum level it allows fermionic solutions by this construction only for isolated values of the coupling constant λ whereas classically $\Psi(x) = Z\, e^{i\lambda \int_{-\infty}^{x} dx' j(x')}$ solves (2b) for any λ. This feature can certainly not be seen by any power expansion in λ.

The current (2c) has been constructed with the bare fermions ψ and since (2c) is sensitive to the infinite renormalization in the dressed field Ψ it is better to replace (2c) by the requirement that j_f is the generator of a local gauge transformation. Indeed,

$$e^{ij_f} \Psi_g e^{-ij_f} = \Psi_{e^{if} g} \qquad (5)$$

holds and in this sense (2c) is also satisfied.

2 The CAR-algebra, its KMS-states and associated v. Neumann algebras

We start with the operator-valued distributions $\psi(x)$, $x \in \mathbf{R}$ which satisfy

$$[\psi^*(x), \psi(x')]_+ = \delta(x-x'). \qquad (6)$$

For $f \in L^2(\mathbf{R})$ we define the bounded operators

$$\psi_f = \int_{-\infty}^{\infty} dx \psi(x) f(x) = \int_{-\infty}^{\infty} \frac{dp}{2\pi} \widetilde{\psi}(p) \widetilde{f}(p),$$

$$\widetilde{f}(p) = \int_{-\infty}^{\infty} dx\, e^{ipx} f(x) \qquad (7)$$

which form a C*-algebra $\mathcal{A}$ characterized by

$$[\psi_f^*, \psi_g]_+ = \langle f|g\rangle = \int dx f^*(x)g(x). \tag{8}$$

We are interested in the automorphisms translation τ_t and parity P and the antiautomorphism charge conjugation C:

$$\tau_t\psi_f = \psi_{f_t}, \quad f_t(x) = f(x-t), \quad P\psi_f = \psi_{Pf}, \\ Pf(x) = f(-x), \quad C\psi_f = \psi_f^*. \tag{9}$$

$\mathcal{A}$ inherits the norm from $L^2(\mathbf{R})$ such that τ_t is (pointwise) normcontinuous in t and even normdifferentiable for the dense set of f's for which

$$\lim_{\delta\downarrow 0}\frac{f(x+\delta)-f(x)}{\delta} = f'(x)$$

exists in $L^2(\mathbf{R})$

$$\frac{d}{dt}\,\tau_t\psi_f\bigg|_{t=0} = -\psi_{f'}. \tag{10}$$

The τ-KMS-states over $\mathcal{A}$ are given by

$$\begin{aligned}\omega_\beta(\psi_f^*\psi_g) &= \int_{-\infty}^{\infty}\frac{dp}{2\pi}\frac{\widetilde{f}^*(p)\widetilde{g}(p)}{1+e^{\beta p}} \\ &= \sum_{n=-\infty}^{\infty}\frac{(-1)^n}{2\pi}\int\frac{dxdx' f^*(x)g(x')}{i(x-x')-n\beta+\varepsilon}, \\ &\quad \varepsilon\downarrow 0, \qquad (11)\\ \omega_\beta(\psi_g\psi_f^*) &= \omega_\beta(\psi_f^*\tau_{i\beta}\psi_g).\end{aligned}$$

With each ω_β are associated a representation π_β with cyclic vector $|\Omega\rangle$, $\omega(a) = \langle\Omega|a|\Omega\rangle$ in $\mathcal{H}_\beta = \overline{\mathcal{A}|\Omega\rangle}$ and a v. Neumann algebra $\pi_\beta(\mathcal{A})''$. It contains the current algebra $\mathcal{A}_c$ which gives the formal expression $j(x) = \psi^*(x)\psi(x)$ a precise meaning. We first observe

Lemma (1)
If the kernel $K(k,k') : \mathbf{R}^2 \to C$ is as operator ≥ 0 and trace class ($K(k,k) \in L^1(\mathbf{R})$), then $\forall\,\beta \in \mathbf{R}^+$

$$\begin{aligned}\lim_{M\to\pm\infty} B_M &:= \lim_{M\to\pm\infty}\frac{1}{(2\pi)^2}\int dkdk' K(k,k') \\ &\quad \times\widetilde{\psi}^*(k+M)\widetilde{\psi}(k'+M) \\ &= \frac{1}{(2\pi)^2}\int dkdk' \lim_{M\to\pm\infty} K(k,k') \\ &\quad \times\omega_\beta(\widetilde{\psi}^*(k+M)\widetilde{\psi}(k'+M)) \\ &= \begin{cases}\frac{1}{2\pi}\int dk\, K(k,k) & \text{for } M\to+\infty \\ 0 & \text{for } M\to-\infty\end{cases}\end{aligned}$$

in the strong sense in $\mathcal{H}_\beta$.

Remarks (1)

1. Lemma (1) substantiates the feeling that for $k > 0$ most levels are empty and for $k < 0$ most are full.
2. B_M is a positive operator and by diagonalizing K one sees
$$\|B_M\| = \|K\|_1 = \frac{1}{2\pi}\int dk\, K(k,k).$$

Proof. Since the norms of B_M are bounded uniformly for all M, it is sufficient to show strong convergence on a dense set in $\mathcal{H}_\beta$. Furthermore

$$\begin{aligned}\|A_M a|\Omega\rangle\|^2 &= \langle\Omega|A_M^* A_M a\tau_{i\beta}a^*|\Omega\rangle \\ &\leq \|A_M\Omega\rangle\|\|A_M a\tau_{i\beta}a^*|\Omega\rangle\| \quad \forall a\in\mathcal{A}.\end{aligned}$$

Thus if $\|A_M|\Omega\rangle\| \to 0$ and $\|A_M\|$ uniformly bounded, then $A_M \to 0$ since with $a \in \mathcal{A}$, $\|a\tau_{i\beta}a^*|\Omega\rangle\| < \infty$ are dense in $\mathcal{H}_\beta$. Now

$$\langle\Omega|(B_M - \langle B_M\rangle)^2|\Omega\rangle = \langle\Omega|B_M^2|\Omega\rangle - \langle\Omega|B_M|\Omega\rangle^2$$

contains the distributions

$$\begin{aligned}&\langle\Omega|\widetilde{\psi}(k+M)^*\widetilde{\psi}(k'+M)\widetilde{\psi}(q'+M)^*\widetilde{\psi}(q+M)|\Omega\rangle \\ &\quad -\langle\Omega|\cdot|\Omega\rangle\,\langle\Omega|\cdot|\Omega\rangle \\ &\quad = \frac{(2\pi)^2\delta(k-q)\delta(k'-q')}{(1+e^{\beta(k+M)})(1+e^{-\beta(k'+M)})}.\end{aligned}$$

This gives for the operators

$$\begin{aligned}&\langle\Omega|B_M^2|\Omega\rangle - \langle\Omega|B_M|\Omega\rangle^2 \\ &\quad = \frac{1}{(2\pi)^2}\int\frac{dkdk'|K(k,k')|^2}{(1+e^{\beta(k+M)})(1+e^{-\beta(k'+M)})}.\end{aligned} \tag{12}$$

Since the Hilbert–Schmidt norm $\int K^2 < \infty$ is less than the trace norm and the integrand in (12) for $M \to \pm\infty$ goes to zero uniformly on compact sets we have established $B_M \to \langle B_M\rangle$ for $M \to \pm\infty$.

If $\int |K|^2$ keeps increasing with M, then $B_M - \langle B_M\rangle$ may nevertheless tend to an (unbounded) operator.

Lemma (2)
If

$$\begin{aligned}B_M = \frac{1}{(2\pi)^2}&\int dkdk'\widetilde{f}(k-k')\Theta(M-|k|)\Theta(M-|k'|) \\ &\times\widetilde{\psi}^*(k)\psi(k')\end{aligned}$$

with $\widetilde{f}$ decreasing faster than an exponential and being the Fourier transform of a positive function, the $B_M - \omega_\beta(B_M)$ is a strong Cauchy sequence for $M \to \infty$ on a dense domain on $\mathcal{H}_\beta$.

Remarks (2)

1. From Remarks (1) we know that $\|B_M\| < 2M\widetilde{f}(0)$ and $f(x) \geq 0$ is not a serious restriction since any function is a linear combination of positive functions.
2. Since the limit j_f is unbounded the convergence is not on all of $\mathcal{H}_\beta$, however since for the limit j_f holds $\tau_{i\beta}j_f = j_{e^{\beta p}f}$, the dense domain is invariant under j_f. Thus we have strong resolvent convergence which means that bounded functions of B_M converge strongly. Also the commutator of the limit is the limit of the commutators.

Proof. As before

$$\langle\Omega|(B_{M'} - B_M - \omega(B_{M'} - B_M))^2|\Omega\rangle$$
$$= \int_{-\infty}^{\infty} \frac{dkdk'}{(2\pi)^2} |\widetilde{f}(k-k')|^2 (1+e^{\beta k})^{-1}(1+e^{-\beta k'})^{-1}$$
$$\times [\Theta(M'-|k|)\Theta(M'-|k'|) -\Theta(M-|k|)\Theta(M'-|k'|)]$$

for $M' > M$. Now with $q = k' - k$ we have

$$\int_M^{M'} \frac{dk}{(1+e^{\beta k})(1+e^{-\beta(k+q)})} \leq \int_M^{M'} dk\, e^{-\beta k}$$

and similarly for $\int_{-M'}^{-M} dk$. Altogether we get

$$\leq \int \frac{dq}{2\pi} |f(q)|^2 \frac{1+e^{\beta|q|}}{2} (e^{-\beta M} - e^{-\beta M'}).$$

By assumption $\int dq < \infty$ thus $\forall\, \varepsilon > 0\ \exists\, M$ such that this is $< \varepsilon\ \forall M' > M$.

We conclude that the limit exists and is selfadjoint on a suitable domain. We shall write it formally

$$j_f = \int_{-\infty}^{\infty} \frac{dkdk'}{(2\pi)^2} \widetilde{f}(k-k') : \widetilde{\psi}(k)^* \widetilde{\psi}(k') : \qquad (13)$$

Next we show that the currents so defined satisfy the CCR with a suitable symplectic form σ [3,10].

Theorem (1)

$$[j_f, j_g] = i\sigma(f,g) = \int_{-\infty}^{\infty} \frac{dp}{(2\pi)^2}\, p\widetilde{f}(p)\widetilde{g}(-p)$$
$$= \frac{i}{4\pi} \int_{-\infty}^{\infty} dx (f'(x)g(x) - f(x)g'(x)).$$

Proof. For the distributions $\widetilde{\psi}(k)$ we get algebraically

$$[\widetilde{\psi}^*(k)\widetilde{\psi}(k'), \widetilde{\psi}^*(q)\widetilde{\psi}(q')] = 2\pi \Big[\widetilde{\psi}^*(k)\widetilde{\psi}(q')\delta(q-k') -\widetilde{\psi}^*(q)\widetilde{\psi}(k')\delta(k-q')\Big]$$

and for the operators after some change of variables

$$\frac{1}{(2\pi)^3} \int dkdpdp' \widetilde{f}(p)\widetilde{g}(p')\widetilde{\psi}^*(p+p'+k)\widetilde{\psi}(k)$$
$$\times\Theta(M-|k|)\Theta(M-|p+p'+k|)$$
$$\times [\Theta(M-|p'+k|) - \Theta(M-|p+k|)].$$

For fixed p and p' and $M \to \infty$ we see that the allowed region for k is contained in $(M-|p|-|p'|, M)$ and $(-M, -M+|p|+|p'|)$. Upon $k \to k \pm M$ we are in the situation of Lemma (1), thus we see that the commutator of the currents (13) is bounded uniformly in M if $\widetilde{f}$ and $\widetilde{g}$ decay faster than exponentials and converges to the expectation value. This gives finally

$$\int_{-\infty}^{\infty} \frac{dp}{(2\pi)^2}\, \widetilde{f}(p)\widetilde{g}(-p) \int dk\, \Theta(M-|k|)\, [\Theta(M-|k-p|) -\Theta(M-|k+p|)]\, \frac{1}{1+e^{\beta k}}$$
$$\stackrel{M\to\infty}{\longrightarrow} \int_{-\infty}^{\infty} \frac{dp}{(2\pi)^2}\, p\widetilde{f}(p)\widetilde{g}(-p).$$

Remarks (3)

1. Since the j_f's satisfy the CCR they cannot be bounded and it is better to write (13) in the Weyl form for the associated unitaries
$$e^{ij_f}\, e^{ij_g} = e^{\frac{i}{2}\sigma(g,f)}\, e^{ij_{f+g}} = e^{i\sigma(g,f)}\, e^{ij_g}\, e^{ij_f}.$$
2. Since j_f is selfadjoint, $e^{i\alpha j_f}$ generate 1–parameter groups. They are the local gauge transformations
$$e^{-i\alpha j_f}\, \psi_g\, e^{i\alpha j_f} = \psi_{e^{i\alpha f} g}.$$
3. The state ω_β can be extended to $\bar{\omega}_\beta$ over $\pi_\beta(\mathcal{A})''$ and τ_t to $\bar{\tau}_t$, $\bar{\tau}_t \in$ Aut $\pi_\beta(\mathcal{A})''$ with $\bar{\tau}_t\, j_f = j_{f_t}$. Furthermore $\bar{\omega}_\beta$ is $\bar{\tau}$–KMS and is calculated to be (Appendix A, see also [11])
$$\bar{\omega}_\beta(e^{ij_f}) = \exp\left[-\frac{1}{2}\int_{-\infty}^{\infty} \frac{dp}{(2\pi)^2} \frac{p}{1-e^{-\beta p}} |\widetilde{f}(p)|^2\right].$$
4. $\bar{\omega}_\beta$ is not invariant under the parity P (9). This symmetry is destroyed in π_β,
$$[j(x), j(x')] = -\frac{i}{2\pi}\, \delta'(x-x')$$
is not invariant under $j(x) \to j(-x)$. Thus $P \notin$ Aut $\pi_\beta(\mathcal{A})''$.
5. The extended shift automorphism $\bar{\tau}_t$ is not only strongly continuous but for suitable f's also differentiable in t (strongly on a dense set in $\mathcal{H}_\beta$)
$$\frac{1}{i}\frac{d}{dt}\bar{\tau}_t e^{ij_f} = \left[j_{f_t'} + \frac{1}{2}\sigma(f_t, f_t')\right] e^{ij_{f_t}}$$
$$= e^{ij_{f_t}} \left[j_{f_t'} - \frac{1}{2}\sigma(f_t, f_t')\right]$$
$$= \frac{1}{2}\left[j_{f_t'}\, e^{ij_{f_t}} + e^{ij_{f_t}} j_{f_t'}\right].$$
6. The symplectic structure is formally independent on β, however for $\beta < 0$ it changes its sign, $\sigma \to -\sigma$, and for $\beta = 0$ (the tracial state) it becomes zero.

3 Extensions of $\mathcal{A}_c$

So far $\mathcal{A}_c$ was defined for j_f's with $f \in C_0^\infty$, for instance. The algebraic structure is determined by the symplectic form $\sigma(f,g)$ () which is actually well defined also for the Sobolev space, $\sigma(f,g) \to \sigma(\bar{f},\bar{g})$, $\bar{f},\bar{g} \in H_1$, $H_1 = \{f : f, f' \in L^2\}$. Also $\bar{\omega}_\beta$ can be extended to H_1, since $\bar{\omega}_\beta(e^{ij_{\bar{f}}}) > 0$ for $\bar{f} \in H_1$. However the anticommuting operators we are looking for are of the form e^{ij_f}, $f(x) = 2\pi\Theta(x_0 - x)$ and though one can give $\sigma(f,g)$ a meaning for such an f, one has in ω_β a divergence for $p \to 0$ and $p \to \infty$

$$\omega_\beta(e^{ij_\Theta}) = \exp\left[-\frac{1}{2}\int_{-\infty}^{\infty} \frac{dp}{p(1-e^{-\beta p})}\right] = 0$$

and thus $\langle f|e^{ij_\Theta}|f\rangle = 0$, where $|f\rangle = e^{ij_f}|\Omega\rangle$ are total in $\mathcal{H}_\beta$. Thus this operator acts as zero in $\mathcal{H}_\beta$. If one tries to approximate Θ by functions from H_1, the unitaries converge weakly to zero.

Example (1)
Denote

$$\varphi_\varepsilon(x) := \begin{cases} 1 & \text{for } x \le -\varepsilon \\ -x/\varepsilon & \text{for } -\varepsilon \le x \le 0 \\ 0 & \text{for } x \ge 0 \end{cases},$$

$$\Phi_{\delta,\varepsilon}(x) := \varphi_\varepsilon(x) - \varphi_\varepsilon(x+\delta) \in H_1,$$

$$\lim_{\substack{\delta\to\infty \\ \varepsilon\to 0}} \Phi_{\delta,\varepsilon}(x) = \Theta(x).$$

Then

$$\widetilde{\Phi}_{\delta,\varepsilon}(p) = \frac{1-e^{ip\varepsilon}}{\varepsilon p^2}(1-e^{ip\delta})$$

and

$$\|\Phi_{\delta,\varepsilon}\|_\beta^2 = \int_{-\infty}^{\infty} \frac{dp}{2\pi}\frac{p}{1-e^{-\beta p}}|\widetilde{\Phi}(p)|^2$$
$$= 16\int_{-\infty}^{\infty} \frac{dp}{2\pi}\frac{p}{1-e^{-\beta p}}\frac{\sin^2 p\varepsilon/2}{\varepsilon^2 p^4}\sin^2 p\delta/2,$$

$$\|\Phi\|_\beta^2 \ge c\int_0^{1/\delta} dp\, \delta^2 = c\delta$$

for β/δ, $\varepsilon/\delta \ll 1$ and c a constant. Thus for $\delta \to \infty$, $\|\Phi_\delta\|_\beta \to \infty$. Also $\|\Phi_\delta - f\|_\beta \to \infty$ since

$$\|\Phi_\delta - f\|_\beta \ge \|\Phi_\delta\|_\beta - \|f\|_\beta \to \infty \qquad \forall\, \|f\|_\beta < \infty$$

and thus

$$|\langle\Omega|e^{-ij_f}\, e^{ij_{\Phi_\delta}}|\Omega\rangle| = e^{-\frac{1}{2}\|\Phi_\delta - f\|_\beta^2} \to 0.$$

But $e^{ij_f}|\Omega\rangle$, $\|f\|_\beta < \infty$, is total in $\mathcal{H}_\beta$ and thus $e^{ij_{\Phi_\delta}}|\Omega\rangle$ and therefore $e^{ij_{\Phi_\delta}}$ goes weakly to zero. However the automorphism

$$e^{ij_f} \to e^{-ij_{\Phi_\delta}} e^{ij_f} e^{ij_{\Phi_\delta}} = e^{i\sigma(\Phi_\delta,f)} e^{ij_f}$$

converges since

$$\begin{aligned}\sigma(f,\Phi_\delta) &= -\frac{1}{2\pi\varepsilon}\left(\int_{-\varepsilon}^{0} - \int_{-\varepsilon-\delta}^{-\delta}\right) dx\, f(x) \\ &\overset{\delta\to\infty}{\longrightarrow} -\frac{1}{2\pi\varepsilon}\int_{-\varepsilon}^{0} dx\, f(x) \\ &\overset{\varepsilon\to 0}{\longrightarrow} -\frac{1}{2\pi} f(0).\end{aligned}$$

This divergence of $\|\Phi_{\delta,\varepsilon}\|$ is related to the well–known infrared problem of the massless scalar field in (1+1) dimensions and various remedies have been proposed [12]. We take it as a sign that one should enlarge $\mathcal{A}_c$ to some $\bar{\mathcal{A}}_c$ and work in the Hilbert space $\bar{\mathcal{H}}$ generated by $\bar{\mathcal{A}}_c$ on the natural extension of the state. Thus we add to $\mathcal{A}_c$ the idealized element $e^{i2\pi j_{\varphi_\varepsilon}} = U_\pi$ and keep σ and ω_β as before. Equivalently we take the automorphism γ generated by U_π and consider the crossed product $\bar{\mathcal{A}}_c = \mathcal{A}_c \overset{\gamma}{\rtimes} \mathbf{Z}$. There is a natural extension $\bar{\omega}$ to $\bar{\mathcal{A}}_c$ and a natural isomorphism of $\bar{\mathcal{H}}$ and $\bar{\mathcal{A}}_c|\bar{\Omega}\rangle$. Here $\bar{\mathcal{H}}$ is the countable orthogonal sum of sectors with n particles created by U_π. Thus,

$$\langle\Omega|e^{ij_f}U_\pi|\Omega\rangle = 0 \tag{14}$$

means that U_π leads to the one-particle sector, in general

$$\langle\Omega|U_\pi^{*n}\, e^{ij_f}U_\pi^m|\Omega\rangle = \delta_{nm}\,\omega_\beta(\gamma^n\, e^{ij_f}).$$

The quasifree automorphisms on $\mathcal{A}_c$ (e.g. τ_t) can be naturally extended to $\bar{\mathcal{A}}_c$, $\tau_t U_\pi = e^{i2\pi j_{\varphi_{\varepsilon,t}}}$, $\varphi_{\varepsilon,t}(x) = \varphi_\varepsilon(x+t)$ and since $\varphi_\varepsilon - \varphi_{\varepsilon,t} \in H_1\ \forall\, t$, this does not lead out of $\bar{\mathcal{A}}_c$.

U_π has some features of a fermionic field since

$$\begin{aligned}\sigma(\varphi_\varepsilon, \tau_t\varphi_\varepsilon) &= -\sigma(\varphi_\varepsilon, \tau_{-t}\varphi_\varepsilon) \\ &= \frac{1}{4\pi}\begin{cases} 1 & \text{for } t > \varepsilon \\ \frac{2t}{\varepsilon} - \frac{t^2}{\varepsilon^2} & \text{for } 0 \le t \le \varepsilon \end{cases}.\end{aligned} \tag{15}$$

More generally we could define $U_\alpha = e^{i\sqrt{2\pi\alpha}j_{\varphi_\varepsilon}}$ and get from (15) with

$$\mathrm{sgn}(t) = \Theta(x) - \Theta(-x) = \begin{cases} 1 & \text{for} \quad t > 0 \\ 0 & \text{for} \quad t = 0 \\ -1 & \text{for} \quad t < 0. \end{cases}$$

Proposition (1)

$$U_\alpha \tau_t U_\alpha = \tau_t(U_\alpha)U_\alpha\, e^{i\,\alpha\,\mathrm{sgn}(t)/2},$$
$$U_\alpha^* \tau_t U_\alpha = \tau_t(U_\alpha)U_\alpha^*\, e^{i\,\alpha\,\mathrm{sgn}(t)/2} \quad \forall\, |t| > \varepsilon.$$

Remark (4)
We note a striking difference between the general case of anyon statistics and the two particular cases – Bose ($\alpha = 2\cdot 2n\pi$) or Fermi ($\alpha = 2(2n+1)\pi$) statistics. Only in the latter two cases parity P (9) is an automorphism of the extended algebra generated through U_α. Thus P which was destroyed in $\mathcal{A}_c$ is now recovered for two subalgebras. The particle sectors are orthogonal in any case

$$\langle\Omega|U^{*n}_\alpha\, e^{ij_f}U_\alpha^m|\Omega\rangle = 0 \quad \forall\, n \ne m,\ f \in H_1.$$

Furthermore, sectors with different statistics are orthogonal $\langle\Omega|U_\alpha^* U_{\alpha'}|\Omega\rangle = 0$, $\alpha \ne \alpha'$, thus if we adjoin $U_\alpha, \forall\alpha \in \mathbf{R}$, $\bar{\mathcal{H}}_\beta$ becomes nonseparable.

Next we want to get rid of the ultraviolet cut–off and let ε go to zero. Proceeding the same way we can extend

σ and τ_t but keeping ω the sectors abound. The reason is that $\varphi_\varepsilon \stackrel{\varepsilon\to 0}{\longrightarrow} \Theta(x)$ and

$$\|\Theta - \Theta_t\|^2 = \int_{-\infty}^{\infty} \frac{dp\, p}{1-e^{-\beta p}} \frac{|1-e^{itp}|^2}{p^2}$$

is finite near $p=0$ but diverges logarithmically for $p \to \infty$. This means that $e^{ij_f} e^{ij_\Theta}|\Omega\rangle$, $f \in H_1$ gives a sector where one of these particles (fermions, bosons or anyons) is at the point $x=0$ and is orthogonal to $e^{ij_f} e^{ij_{\Theta_t}}|\Omega\rangle\ \forall\, t \neq 0$. Thus the total Hilbert space is not separable and the shift τ_t is not even weakly continuous. Thus there is no chance to make sense of $\frac{d}{dt}\tau_t e^{ij_\Theta}$.

4 Anyon fields in $\pi_{\bar{\omega}}(\bar{\mathcal{A}}_c)''$

Next we shall use another ultraviolet limit to construct local fields which obey some anyon statistics. Of course quantities like

$$\begin{aligned}[\Psi^*(x), \Psi(x')]_\alpha &:= \Psi^*(x)\Psi(x') e^{i\frac{2\pi-\alpha}{4}\mathrm{sgn}(x'-x)} \\ &\quad + \Psi(x')\Psi^*(x) e^{-i\frac{2\pi-\alpha}{4}\mathrm{sgn}(x'-x)} \\ &= \delta(x-x')\end{aligned}$$

will only be operator valued distributions and have to be smeared to give operators. Furthermore in this limit the unitaries we used so far have to be renormalized so that $\delta(x-x')$ gets a factor 1 in front. A candidate for $\Psi(x)$ will be ($\alpha \in (0, 4\pi)$)

$$\Psi(x) := \lim_{\varepsilon\to 0} n(\varepsilon) \exp\left[i\sqrt{2\pi\alpha}\int_{-\infty}^{\infty} dy\ \varphi_\varepsilon(x-y) j(y)\right]$$

with φ_ε from the example if Sect. 3 and $n(\varepsilon)$ a suitably chosen normalization. With the shorthand $\varphi_{\varepsilon,x}(y) = \varphi_\varepsilon(x-y)$ we can write

$$\begin{aligned}\Psi^*_\varepsilon(x)\Psi_\varepsilon(x') &= \exp\{i\,2\pi\alpha\,\sigma(\varphi_{\varepsilon,x}, \varphi_{\varepsilon,x'})\} \\ &\quad \times \exp\left\{i\sqrt{2\pi\alpha}\, j_{\varphi_{\varepsilon,x'}-\varphi_{\varepsilon,x}}\right\}, \\ \Psi_\varepsilon(x')\Psi^*_\varepsilon(x) &= \exp\{-i\,2\pi\alpha\,\sigma(\varphi_{\varepsilon,x}, \varphi_{\varepsilon,x'})\} \\ &\quad \times \exp\left\{i\sqrt{2\pi\alpha}\, j_{\varphi_{\varepsilon,x'}-\varphi_{\varepsilon,x}}\right\}.\end{aligned}$$

We had in (15)

$$\begin{aligned}4\pi\sigma(\varphi_{\varepsilon,x}, \varphi_{\varepsilon,x'}) &= \mathrm{sgn}(x-x')\Big\{\Theta(|x-x'|-\varepsilon) \\ &\quad + \Theta(\varepsilon - |x-x'|)\frac{(x-x')^2}{\varepsilon^2}\Big\} \\ &=: \mathrm{sgn}(x-x') D_\varepsilon(x-x')\end{aligned}$$

and thus

$$\begin{aligned}&[\Psi^*_\varepsilon(x), \Psi_\varepsilon(x')]_\alpha = 2n(\varepsilon)^2 \\ &\quad \times \cos\left[\mathrm{sgn}(x-x')\left(\frac{\pi}{2} - \frac{\alpha}{4}(1 - D_\varepsilon(x-x'))\right)\right] \\ &\quad \times \exp\left[i\sqrt{2\pi\alpha} j_{\varphi_{\varepsilon,x'}-\varphi_{\varepsilon,x}}\right].\end{aligned}$$

Note that for $|x-x'| \geq \varepsilon$ the argument of the cos becomes $\pm\pi/2$, so the α–commutator vanishes, in agreement with Proposition (1). To manufacture a δ-function for $|x-x'| \leq \varepsilon$ we note that $cos(...) > 0$ and $\omega_\beta(e^{i\alpha j}) > 0$, so we have to choose $n(\varepsilon)$ such that

$$\begin{aligned}&2n^2(\varepsilon)\varepsilon \int_{-1}^{1} d\delta \cos\left(\frac{\pi}{2} - \frac{\alpha}{4}(1-\delta^2)\right) \\ &\quad \times \omega_\beta\left(\exp\left[i\sqrt{2\pi\alpha} j_{\varphi_{\varepsilon,x-\varepsilon\delta}-\varphi_{\varepsilon,x}}\right]\right) = 1\end{aligned}$$

and to verify that for $\varepsilon \downarrow 0$ $[\,]_\alpha$ converges strongly to a c-number. For the latter to be finite we have to smear $\Psi(x)$ with L^2-functions g and h:

$$\begin{aligned}&\int dx dx' g^*(x) h(x') [\Psi^*_\varepsilon(x), \Psi_\varepsilon(x')]_\alpha \\ &= \int dx dx' g^*(x) h(x') 2n(\varepsilon)^2 \\ &\quad \times \cos(\ )\exp\left[i\sqrt{2\pi\alpha} j_{\varphi_{\varepsilon,x'}-\varphi_{\varepsilon,x}}\right].\end{aligned}$$

This converges strongly to $\langle g|h\rangle$ if for $\varepsilon \downarrow 0$

$$\begin{aligned}&\left\langle \exp\left[-i\sqrt{2\pi\alpha} j_{\varphi_{\varepsilon,x'}-\varphi_{\varepsilon,x}}\right] \exp\left[i\sqrt{2\pi\alpha} j_{\varphi_{\varepsilon,y'}-\varphi_{\varepsilon,y}}\right]\right\rangle \\ &- \left\langle \exp\left[-i\sqrt{2\pi\alpha} j_{\varphi_{\varepsilon,x'}-\varphi_{\varepsilon,x}}\right]\right\rangle \left\langle \exp\left[i\sqrt{2\pi\alpha} j_{\varphi_{\varepsilon,y'}-\varphi_{\varepsilon,y}}\right]\right\rangle \to 0\end{aligned}$$

for almost all x, x', y, y'. Now

$$\langle e^{-ij_a} e^{ij_b}\rangle = \langle e^{-ij_a}\rangle\langle e^{ij_b}\rangle \exp\left[\int_{-\infty}^{\infty} \frac{dp\, p}{1-e^{\beta p}} \widetilde{a}(-p)\widetilde{b}(p)\right].$$

In our case this last factor is

$$\begin{aligned}&\int_{-\infty}^{\infty} \frac{dp\, p}{1-e^{-\beta p}} \frac{|1-e^{ip\varepsilon}|^2}{\varepsilon^2 p^4} (e^{ipx} - e^{ipx'})(e^{-ipy} - e^{-ipy'}) \\ &= \int_{-\infty}^{\infty} \frac{dp\ 2(1-\cos p)}{p^3(1-e^{-\beta p/\varepsilon})} (e^{ipx/\varepsilon} - e^{ipx'/\varepsilon}) \\ &\quad \times (e^{-ipy/\varepsilon} - e^{-ipy'/\varepsilon}).\end{aligned}$$

For fixed $\beta \neq 0$ and almost all x, x', y, y' this converges to zero for $\varepsilon \to 0$ by Riemann-Lebesgue. In the same way one sees that $\exp\left[i\sqrt{2\pi\alpha} j_{\varphi_{\varepsilon,x}+\varphi_{\varepsilon,x'}}\right]$ converges strongly to zero and that the $\Psi_{\varepsilon,g}$ are a strong Cauchy sequence for $\varepsilon \to 0$. To summarize we state

Theorem (2)
$\Psi_{\varepsilon,g}$ converges strongly for $\varepsilon \to 0$ to an operator Ψ_g which for $\alpha = 2\pi$ satisfies

$$[\Psi^*_g, \Psi_h]_+ = \langle g|h\rangle, \qquad [\Psi_g, \Psi_h]_+ = 0.$$

If *supp* $g <$ *supp* h,

$$\Psi^*_g \Psi_h\, e^{i\frac{2\pi-\alpha}{4}} + \Psi_h \Psi^*_g\, e^{-i\frac{2\pi-\alpha}{4}} = 0 \qquad \forall \alpha.$$

Furthermore we have to verify the claim (5) that also for Ψ_g the current j_f induces the local gauge transformation $g(x) \to e^{2i\alpha f(x)} g(x)$. For finite ε we have

$$e^{ij_f} \Psi_{\varepsilon,g} e^{-ij_f} = \Psi_{\varepsilon, e^{i2\pi\alpha\sigma(f,\varphi_\varepsilon)} g}$$

and for $\varepsilon \downarrow 0$ we get $\sigma(f, \varphi_\varepsilon) \to \frac{1}{2\pi} f(0)$, so that $\sigma(f, \tau_x \varphi_\varepsilon) = \frac{1}{2\pi} f(x)$.

To conclude we investigate the status of the "Urgleichung" in our construction. It is clear that the product of operator valued distributions on the r.h.s. can assume a meaning only by a definite limiting prescription. Formally it would be

$$\begin{aligned} \Psi(x)\Psi^*(x)\Psi(x) &= [\Psi(x), \Psi^*(x)]_+ \Psi(x) - \Psi^*(x)\Psi(x)^2 \\ &= \delta(0)\Psi(x) - 0. \end{aligned}$$

From Theorem (1) and (10) we know

$$\begin{aligned} \frac{1}{i}\frac{\partial}{\partial x}\Psi_\varepsilon(x) &= \frac{\sqrt{2\pi\alpha}}{2}\, [\bar{\jmath}(x), \Psi_\varepsilon(x)]_+, \\ \bar{\jmath}(x) &= \frac{1}{\varepsilon}\int_{x-\varepsilon}^{x} dy\, j(y). \end{aligned}$$

Using $j_{\varphi'} e^{ij_\phi} = \frac{1}{i}\frac{\partial}{\partial\alpha} e^{i\frac{\alpha}{2}\sigma(\varphi',\varphi)} e^{ij_{\varphi+\alpha\varphi'}}|_{\alpha=0}$ one can verify that the limit $\varepsilon \downarrow 0$ exists for the expectation value with a total set of vectors and thus gives densely defined (not closable) quadratic forms. They do not lead to operators but we know from Lemma (1) that they define operator valued distributions for test functions from H_1. Thus one could say that in the sense of operator valued distributions the Urgleichung holds

$$\frac{1}{i}\frac{\partial}{\partial x}\Psi(x) = \frac{\sqrt{2\pi\alpha}}{2}\, [j(x), \Psi(x)]_+. \tag{16}$$

The remarkable point is that the coupling constant λ in (1) is related to the statistics parameter α. For fermions one has a solution only for $\lambda = \sqrt{2\pi}$. Of course one could for any λ enforce fermi statistics by renormalizing the bare fermion field $\psi \to \sqrt{Z}\ \psi$, $j \to Zj$ with a suitable $Z(\lambda)$ but this just means pushing factors around. Alternatively one could extend $\mathcal{A}_c$ by adding $e^{i\sqrt{2\pi\alpha}\, j_{\varphi_\varepsilon}}$, for all $\alpha \in \mathbf{R}^+$. Then one gets in $\bar{\mathcal{H}}_\beta$ uncountably many orthogonal sectors, one for each α, and in each sector a different Urgleichung holds. Thus different anyons live in orthogonal Hilbert spaces and $e^{i\sqrt{2\pi\alpha}\, j_{\varphi_\varepsilon}}$ is not even weakly continuous in α. If α is tied to λ it is clear that an expansion in λ is doomed to failure and will never reveal the true structure of the theory.

5 Concluding remarks

To summarize we gave a precise meaning to (2a,b,c) by starting with bare fermions, $\mathcal{A} = \mathrm{CAR}(\mathbf{R})$. The shift τ_t is an automorphism of $\mathcal{A}$ which has KMS–states ω_β and associated representations π_β. In $\pi_\beta(\mathcal{A})''$ one finds bosonic modes $\mathcal{A}_c$ with an algebraic structure independent on β. Taking the crossed product with an outer automorphism of $\mathcal{A}_c$ or equivalently augmenting $\mathcal{A}_c$ by an unitary operator to $\bar{\mathcal{A}}_c$ we discover in $\bar{\pi}_\beta(\mathcal{A}_c)''$ anyonic modes which satisfy the Urgleichung in a distributional sense. For special values of λ they are dressed fermions distinct from the bare ones. From the algebraic inclusions CAR(*bare*) $\subset \pi_\beta(\mathcal{A})'' \supset \mathcal{A}_c \subset \bar{\mathcal{A}}_c \subset \bar{\pi}_\beta(\bar{\mathcal{A}}_c)'' \supset$ CAR(*dressed*) one concludes that in our model it cannot be decided whether fermions or bosons are more fundamental. One can construct the dressed fermions either from bare fermions or directly from the current algebra and our original question remains open like the one whether the egg or the hen was first.

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Appendix A KMS–states – Dirac sea and the Schwinger term

An equilibrium state of a quantum system at finite temperature $T = \beta^{-1}$ is characterized by the KMS–condition

$$\omega_\beta\left(\tau_t(A)\, B\right) = \omega_\beta(B\, \tau_{t+i\beta} A) \tag{A.1}$$

with the time evolution τ_t as an automorphism of the algebra of observables $\mathcal{A}$ analytically continued for imaginary times. Thus, an equilibrium state for a system with an infinite number of free bosons can be defined through the quasifree state over the algebra of smeared creation and annihilation operators a_f^*, a_g,

$$a_f^{(*)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} a^{(*)}(p)\tilde{f}^{(*)}(p)dp$$

so that for the non–smeared operators one has

$$\langle a^*(p)a(k)\rangle = \frac{2\pi\, p\, \delta(p-k)}{1 - e^{-\beta p}} \tag{A.2}$$

similarly for fermions

$$\langle a^*(p)a(k)\rangle = \frac{2\pi\, \delta(p-k)}{1 + e^{\beta p}} \tag{A.3}$$

Note that for (A.3) to be a well defined state there is no need for the Hamiltonian to be bounded from below, in contrast to the $T = 0$ case. There, a Bogoliubov transformation is needed to ensure semiboundedness for the free Hamiltonian. As has been realized already in the thirties [3,4], such a manipulation (corresponding to filling in the Dirac sea) leads to an anomalous term in the current commutator – Theorem (1). One could be therefore misleaded to think that the KMS–state ignores this anomaly. Actually, it is the other way round – the KMS–state automatically takes care for the Dirac vacuum since for negative momenta (A.3) transforms into

$$\langle a(p)a^*(k)\rangle = \frac{2\pi\,\delta(p-k)}{1+e^{-\beta p}}$$

and this corresponds exactly to exchanging the roles of creation and annihilation operators.

Indeed, in momentum space, with

$$\begin{aligned}\rho(p) &= \int_{-\infty}^{\infty}\psi^*(x)\psi(x)e^{ipx}dx\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}a^*(k+p)a(k)dk\\ \rho(-p) &= \int_{-\infty}^{\infty}\psi^*(x)\psi(x)e^{-ipx}dx\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}a^*(k)a(k+p)dk\end{aligned}$$

p always positive, one gets $(:\rho := \rho - \langle\rho\rangle)$

$$\begin{aligned}\langle :\rho(-p)::\rho(p'):\rangle &= \int_{-\infty}^{\infty}\frac{dkdk'}{(2\pi)^2}\langle a^*(k)a(k')\rangle\\ &\quad\times\langle a(k+p)a^*(k'+p')\rangle\\ &= \int_{-\infty}^{\infty}\frac{dk\,\delta(p-p')}{(1+e^{\beta k})\left(1+e^{-\beta(k+p)}\right)}\\ &= \frac{\delta(p-p')}{\beta\,(1-e^{-\beta p})}\ln\frac{1+e^{-\beta k}}{e^{\beta p}+e^{-\beta k}}\Bigg|_{-\infty}^{\infty}\\ &= \frac{p}{1-e^{-\beta p}}\,\delta(p-p')\\ &= F(p)\,\delta(p-p') \qquad\qquad (A.4)\end{aligned}$$

Then with the representation π_β the following KMS–state over the observables algebra $\mathcal{A}_c$ is accociated

$$\omega_\beta(e^{ij_f}) = exp\left\{-\frac{1}{2}\int_{-\infty}^{\infty}\frac{dp}{(2\pi)^2}\,\frac{p}{1-e^{-\beta p}}\,|\tilde{f}(p)|^2\right\}$$

as follows from the general form of KMS–states over a Weyl algebra [13].

Similarly,

$$\begin{aligned}\langle :\rho(p')::\rho(-p):\rangle &= -\frac{p}{1-e^{\beta p}}\,\delta(p-p')\\ &= F(-p)\,\delta(p-p') \qquad\qquad (A.5)\end{aligned}$$

For $F(p)$ the following relation holds

$$F(-p) = e^{-\beta p}\,F(p) > 0 \quad \forall p\in\mathbf{R} \qquad (A.6)$$

With $\tau_t\rho(p) = e^{ipt}\rho(p) \longrightarrow e^{\beta p}\rho(p)$ and (A.6), validity of the KMS–condition, (A.1), is verified

$$\begin{aligned}\langle :\rho(-p)::\tau_{i\beta}\rho(p'):\rangle &= e^{-\beta p}F(p)\delta(p-p') = F(-p)\delta(p-p')\\ &= \langle :\rho(p')::\rho(-p):\rangle\end{aligned}$$

So, (A.4), (A.5) correspond to a KMS–state over a bosonic algebra and are both temperature dependent. This is not the case for the commutator itself

$$\langle[\rho(p),\rho(-p')]\rangle = F(p)\left(1-e^{-\beta p}\right)\delta(p-p') = p\,\delta(p-p')$$

This is the well–known result from the $T=0$ case. Thus, the KMS–state for $\beta > 0$ is by construction associated with the Dirac vacuum and the current anomaly is recovered but it does not depend on the temperature (see also [14]) despite the fact that the correlator functions do.

Appendix B Non–commuting fields through crossed products

The idea that the crossed product C^*–algebra extension is the tool that makes possible construction of fermions (so, unobservable fields) from the observable algebra has been first stated in [15]. There, the problem of obtaining different field groups has been shown to amount to construction of extensions of the observable algebra by the group duals. Explicitly, crossed products of C^*–algebras by semigroups of endomorphisms have been introduced when proving the existence of a compact global gauge group in particle physics given only the local observables [16]. Also in the structural analysis of the symmetries in the algebraic QFT [2] extendibility of automorphisms from a unital C^*–algebra to its crossed product by a compact group dual becomes of importance since it provides an analysis of the symmetry breaking [17] and in the case of a broken symmetry allows for concrete conlusions for the vacuum degeneracy [18].

The reason why a relatively complicated object – crossed product over a specially directed symmetric monoidal subcategory End $\mathcal{A}$ of unital endomorphisms of the observable algebra $\mathcal{A}$, is involved in considerations in [18] is that in general, non–Abelian gauge groups are envisaged. For the Abelian group $U(1)$ a significant simplification is possible since its dual is also a group – the group $\mathbf{Z}$. On the other hand, even in this simple case the problem of describing the local gauge transformations remains open. Therefore in the Abelian case consideration of crossed products over a discrete group offers both a realistic framework and reasonable simplification for the analysis of the resulting field algebra. We shall briefly outline the general construction for this case, for more details see [19].

We start with the CCR algebra $\mathcal{A}(\mathcal{V}_0,\sigma)$ over the real symplectic space $\mathcal{V}_0$ with symplectic form σ, (13), generated by the unitaries $W(f)$, $f\in\mathcal{V}_0$ with

$$\begin{aligned}W(f_1)W(f_2) &= e^{i\sigma(f_1,f_2)}W(f_1+f_2),\\ W(f)^* &= W(-f) = W(f)^{-1}.\end{aligned}$$

Instead of the canonical extension $\bar{\mathcal{A}}(\mathcal{V},\bar{\sigma})$, $\mathcal{V}_0\subset\mathcal{V}$ [9], we want to construct another algebra $\mathcal{F}$, such that CCR($\mathcal{V}_0$)

$\subset \mathcal{F} \subset \mathrm{CCR}(\mathcal{V})$ and we choose $\mathcal{V}_0 = \mathcal{C}_0^\infty$, $\mathcal{V} = \partial^{-1}\mathcal{C}_0^\infty$. Any free (not inner) automorphism α, $\alpha \in \mathrm{Aut}\,\mathcal{A}$ defines a crossed product $\mathcal{F} = \mathcal{A} \overset{\alpha}{\bowtie} \mathbf{Z}$. This may be thought as (see [20]) adding to the initial algebra $\mathcal{A}$ a single unitary operator U together with all its powers, so that one can formally write $\mathcal{F} = \sum_n \mathcal{A} U^n$, with U implementing the automorphism α in $\mathcal{A}$, $\alpha A = U A U^*$, $\forall A \in \mathcal{A}$. Operator U should be thought of as a charge–creating operator and $\mathcal{F}$ is the minimal extension – an important point in comparison to the canonical extension which we find superfluous, especially when questions about statistical behaviour and time evolution are to be discussed. With the choice

$$\alpha W(f) = e^{i\sigma(\bar{g},f)} W(f), \quad \bar{g} \in \mathcal{V}\backslash\mathcal{V}_0, \quad \mathcal{V}_0 \subset \mathcal{V} \qquad \text{(B.1)}$$

and identifying $U = W(\bar{g})$, $\mathcal{F}$ is in a natural way a subalgebra of $\mathrm{CCR}(\mathcal{V})$.

If we take for $\mathcal{A}$ the current algebra $\mathcal{A}_c$ and for U – the idealized element U_π to be added to it, we find an obvious correspondence between the functional picture from Sect. 3 and the crossed product construction. However, in the latter there is an additional structure present which makes it in some cases favourable. Writing an element $F \in \mathcal{F}$ as $F = \sum_n A_n U^n$, $A_n \in \mathcal{A}$, we see that it is convenient to consider $\mathcal{F}$ as an infinite vector space with U^n as its basic unit vectors and $A_n =: (F)_n$ as components of F. The algebraic structure of $\mathcal{F}$ implies that multiplication in this space is not componentwise but instead

$$(F.G)_m = \sum_n F_n \, \alpha^n \, G_{m-n}.$$

Given a quasifree automorphism $\rho \in \mathrm{Aut}\,\mathcal{A}$, it can be extended to $\mathcal{F}$ iff the related automorphism $\gamma_\rho = \rho\alpha\rho^{-1}\alpha^{-1}$ is inner for $\mathcal{A}$. Since γ_ρ is implemented by $W(\bar{g}_\rho - \bar{g})$, this is nothing else but demanding that $\bar{g}_\rho - \bar{g} \in \mathcal{V}_0$ and this is exactly the same requirement as in the functional picture. This appears to be the case for the space translations and also for the time evolution, but in the absence of long–range forces [19].

Also a state $\omega(.)$ over $\mathcal{A}$ together with the representation π_ω associated with it through the GNS–construction can be extended to $\mathcal{F}$. The representation space of $\mathcal{F}$ can be regarded as a direct sum of charge–n subspaces, each of them being associated with a state $\omega \circ \alpha^{-n}$ and with $\mathcal{H}_0$, the representation space of $\mathcal{A}$, naturally imbedded in it. Since ω is irreducible and $\omega \circ \alpha^{-n}$ not normal with respect to it, the extension of the state over $\mathcal{A}$ to a state over $\mathcal{F}$ is uniquely determined by the expectation value with $|\Omega_0\rangle = |\omega\rangle$ in this representation

$$\langle \Omega_k | W^*(f) W(h) W(f) | \Omega_n \rangle = \delta_{kn} \, e^{-i\sigma(f+n\bar{g},h)} \omega(W(h))$$

where $U_k|\Omega\rangle := |\Omega_k\rangle$, $\langle \Omega_k|\Omega_n\rangle = \delta_{kn}$. This states nothing but orthogonality of the different charge sectors, the same as in the functional description, (14).

In the crossed product gauge automorphism is naturally defined with

$$\gamma_\nu \, U^n = e^{2\pi i \nu n} U^n, \qquad \gamma_\nu \, W(f) = W(f) \qquad \text{(B.2)}$$

Thus for the representation π_Ω one finds

$$\gamma_\nu \left(|F(f)^{(k)}\rangle \right) = \gamma_\nu \left(W(f)|\Omega_k\rangle \right) = e^{2\pi i \nu k} W(f)|\Omega_k\rangle,$$

that justifies interpretation of the vectors $|F(f)^{(k)}\rangle$ as belonging to the charge–k subspace. However, $\mathcal{A}$ is a subalgebra of $\mathcal{F}$ for the gauge group $\mathcal{T} = [0,1)$, while it is a subalgebra of CAR for the gauge group $\mathcal{T} \otimes \mathbf{R}$. Thus the crossed product algebra so constructed, being really a Fermi algebra, does not coincide with CAR but is only contained in it. In other words, such a type of extension does not allow incorporation also of local gauge transformations which are of main importance in QFT.

Therefore we need a generalization of the construction in [19] which would describe also the local gauge transformations. The most natural candidate for a structural automorphism would be

$$\alpha_{\bar{g}_x} W(f) = e^{i \sum_{n=0}^{K} f^{(n)}(x)} W(f). \qquad \text{(B.3)}$$

However, it turns out that only for $K = 0$ the crossed product algebra so obtained allows for extension of space translations as an automorphism of $\mathcal{A}$ – the minimal requirement one should be able to meet. Already first derivative gives for the zero Fourier component of the difference $\bar{g}_{x_\delta} - \bar{g}_x$ an expression of the type $\int y^{-1}\delta(y)dy$, so it drops out of $\mathcal{C}_0^\infty$. So, the automorphism of interest reads

$$\alpha_{\bar{g}_x} W(f) = e^{if(x)} W(f) \qquad \text{(B.4)}$$

and can be interpreted as being implemented by $W(\bar{g}_x)$ with $\bar{g}_x = 2\pi\,\Theta(x-y)$. Correspondingly, the operator we add to $\mathcal{A}$ through the crossed product is

$$U_x = e^{i2\pi \int_{-\infty}^{x} j(y)dy}. \qquad \text{(B.5)}$$

Compared to [19] this means an enlargment of the test functions space not with a kink but with its limit – the sharp step function. In a distributional sense it still can be considered as an element of $\partial^{-1}\mathcal{V}_0$ for some $\mathcal{V}_0$ since the derivative of $\bar{g}_x$ has bounded zero Fourier component. Similarly, the extendibility condition for space translations is found to be satisfied, $\bar{g}_{x_\delta} - \bar{g}_x \in \mathcal{V}_0$ so that in the crossed product shifts are given by

$$\bar{\tau}_{x_\delta} U_x = V_{x_\delta} U_x, \qquad V_{x_\delta} = W(\bar{g}_{x_\delta} - \bar{g}_x). \qquad \text{(B.6)}$$

Note that shifts do not commute with the structural automorphism $\alpha_{\bar{g}_x}$, $\tau_{x_\delta}\alpha_{\bar{g}_x} W(f) \neq \alpha_{\bar{g}_x}\tau_{x_\delta} W(f)$. Since

$$\sigma(\bar{g}_x, \bar{g}_{x_\delta}) = -\pi \mathrm{sgn}(\delta), \qquad \text{(B.7)}$$

already the elements of the first class are anticommuting and we identify $U_x =: \psi(x)$. Then (B.4) (after smearing with a function from $\mathcal{C}_0^\infty$) is nothing else but (5), i.e. the statement (or requirement) that currents generate local gauge transformations of the so–constructed field. Any scaling of the function which defines the structural automorphism $\alpha_{\bar{g}_x}$ destroys this relation and fields obeying fractional statistics are obtained instead. This is effectively the same as adding to the algebra $\mathcal{A}$ the element U_α with $\alpha = 2\pi\mu$, μ being the scaling parameter.

However, the crossed product offers one more interesting possibility: when for the symplectic form in question instead of (B.7) (or its direct generalization $\sigma(\bar{g}_x, \bar{g}_{x_\delta}) = (2n+1)\pi, n \in \mathbf{Z}$) another relation takes place, $\sigma(\bar{g}_x, \bar{g}_{x_\delta}) = (2n+1)/\bar{n}^2$ for some fixed $\bar{n} \in \mathbf{Z}$, the crossed product acquires a zone structure, with $2n\bar{n}$–classes commuting, $(2n+1)$–classes anticommuting and elements in the classes with numbers $m \in \mathbf{Z}/\mathbf{Z}_{\bar{n}}$ obeying an anyon statistics with parameter $r = \sqrt{2n+1}\, m/\bar{n}$. So, fields with different statistical behaviour are present in the same algebra, however the Hilvbert space remains separable (which would not be the case if non–Abelian group has been considered).

We want to emphasize that relation of the type $\psi(x+\delta_x) = U_{x+\delta_x}$ may be misleading, the latter element exists in the crossed product only by (B.6), so that for the derivative one finds

$$\begin{aligned}\frac{\partial\psi(x)}{\partial x} &:= \lim_{\delta_x\to 0} \frac{\psi(x+\delta_x)-\psi(x)}{\delta_x} \\ &= \lim_{\delta_x\to 0} \frac{1}{\delta_x}\left(V_{x_\delta}U_x - U_x\right) \\ &= \lim_{\delta_x\to 0} \frac{1}{\delta_x}\left(e^{i\,2\pi\,\delta_x\, j(x)} - 1\right) U_x \\ &= 2\pi\, i\, j(x) U_x =: 2\pi\, i\, j(x)\psi(x). \qquad \text{(B.8)}\end{aligned}$$

This, together with (10) gives for the operators

$$i\psi_{f'} = \psi_f\, j_{\Theta'}. \qquad \text{(B.9)}$$

Note that in the crossed product, which can actually be considered as a left $\mathcal{A}$–module, equations of motion (B.8), (B.9) appear (due to this reason) without an antisymmetrization, which was the case with the functional realization, (16), but otherwise the result is the same. Therefore the scaling sensitivity of the crossed product field algebra is another manifestation of the quantum "selection rule" for the value of λ in Urgleichung (2b).

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